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Generalised variational derivatives in field theory

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Abstract. A new class of variational derivatives is used to deduce several algebraic properties of Lie–Bäcklund operators in the context of the variational formalism. Applications to the conservation theorems in the Lagrangian and the Hamiltonian formalisms are given.

1. Introduction

A new class of variational derivatives has been recently introduced (Galindo and Martínez Alonso 1978) which turns out to be very appropriate in studying the kernel and the range of the usual variational derivative. In this paper we use these generalised variational derivatives as a tool to investigate the connection between invariance groups and conservation laws in the Lagrangian and Hamiltonian formalisms.

Our analysis is based on several algebraic properties of the generalised variational derivatives in the context of the theory of groups of Lie–Bäcklund tangent transformations. These properties enable us to express the Lie–Bäcklund operators in a convenient form in order to deal with them in the variational formalism. In particular, a useful identity for the commutator of Lie–Bäcklund operators with the usual variational derivative is obtained. We show in this paper that this identity plays an important role in the derivations of conservation theorems in field theory.

It is well known that invariance properties of systems of partial differential equations may be studied in terms of the theory of Lie–Bäcklund operators (Ibragimov and Anderson 1976, Kumei 1975, 1977, 1978). However there are no general rules for deriving conservation laws from the existence of invariance groups for a given system of partial differential equations. Notwithstanding, in Lagrangian formalism Noether's theorem gives a way of doing this for a certain class of invariance groups. In our opinion the origin of this particular property of Lagrangian systems has not been conveniently explained. In this paper we show that Noether's theorem is a consequence of the algebraic relations between Lie–Bäcklund operators and variational derivatives. Moreover we discuss the advantages of our approach to Noether's result with respect to the conventional one based on the invariance properties of the action functional. On the other hand, we also investigate the applications of the generalised variational derivatives in the Hamiltonian formalism. Thus we find that they appear in the formula governing the time evolution of density functions, and we use them to deduce the commutation relations between variational derivatives and canonical operators. As a consequence we obtain a simple proof of a conservation theorem in the Hamiltonian formalism due to Kumei (1978). We emphasise that our derivation of Kumei's result avoids the use of functionals and uses only algebraic properties of variational derivatives and canonical operators in the context of density functions.

2. Generalised variational derivatives and Lie–Bäcklund operators

Let $u^r (r = 1, \ldots, m)$ be m real fields depending on n coordinates $x_i (i = 1, \ldots, n)$. For each ordered set $\alpha = (\alpha_1, \ldots, \alpha_n)$ of n non-negative integers, we denote

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \qquad u^r_\alpha = \frac{\partial^{|\alpha|} u^r}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Given $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ we shall write $\beta \leqslant \alpha$ if $\beta_i \leqslant \alpha_i$ for all $i = 1, \ldots, n$. In this case we define

$$\binom{\alpha}{\beta} = \prod_{i=1}^{n} \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!}. \tag{1}$$

We shall consider functions $F = F[x, u]$, which depend on a finite number of variables belonging to the set $(x_i, u^r_\alpha)(i = 1, \ldots, n; r = 1, \ldots, m; |\alpha| \geqslant 0)$. From now on we shall limit ourselves to functions $F \in C^\infty$, and we will denote by $\mathscr{F}$ the set of all such F's. The summation rule over repeated indices will be assumed for the r index which specifies the field components.

We define the variational derivative with respect to the variable u^r_α as the following operator on $\mathscr{F}$:

$$\frac{\delta F}{\delta u^r_\alpha} = \sum_\beta (-1)^{|\beta|} \binom{\alpha + \beta}{\beta} D^\beta \frac{\partial F}{\partial u^r_{\alpha+\beta}}, \tag{2}$$

where D^β denotes the total differentiation operator

$$D^\beta = \prod_{i=1}^{n} (D^i)^{\beta_i}, \qquad D^i = \frac{\partial}{\partial x_i} + \sum_\alpha \frac{\partial u^r_\alpha}{\partial x_i} \frac{\partial}{\partial u^r_\alpha}.$$

In particular, $\delta/\delta u^r$ coincides with the usual variational derivative

$$\frac{\delta F}{\delta u^r} = \sum_\beta (-1)^{|\beta|} D^\beta \frac{\partial F}{\partial u^r_\beta}. \tag{3}$$

Given functions $\mu^r = \mu^r[x, u]$ and $F = F[x, u]$ in $\mathscr{F}$, we have the identity (Galindo and Martínez Alonso 1978)

$$\sum_\alpha \frac{\partial F}{\partial u^r_\alpha} D^\alpha \mu^r = \sum_\alpha D^\alpha \left(\frac{\delta F}{\delta u^r_\alpha} \mu^r \right). \tag{4}$$

As a consequence, it follows that every $F \in \mathscr{F}$ admits the expansion

$$F[x, u] = F[x, 0] + \sum_\alpha D^\alpha \int_\gamma \frac{\delta F}{\delta v^r_\alpha} \mathrm{d} v^r, \tag{5}$$

γ being the curve $\gamma(t) = [x, tu] (t \in [0, 1])$. This expansion has been used by Galindo and Martínez Alonso to prove rigorously the important property

$$\ker \delta/\delta u = \operatorname{ran} \boldsymbol{D} \tag{6}$$

where $\ker \delta/\delta u = \{F \in \mathcal{F}: \delta F/\delta u^r = 0\ \forall r\}$ and $\operatorname{ran} \boldsymbol{D}$ is the linear space of all the elements of $\mathcal{F}$ which are of the form $\sum_{i=1}^{n} D^i F_i (F_i \in \mathcal{F})$.

The following proposition shows how the generalised variational derivatives appear in a natural way in the computation of the usual variational derivative of the product of two functions.

Proposition 1.

$$\frac{\delta}{\delta u^r}(FG) = \sum_\alpha (-1)^{|\alpha|}\left(D^\alpha F.\frac{\delta G}{\delta u^r_\alpha} + \frac{\delta F}{\delta u^r_\alpha} D^\alpha G\right).$$

Proof. From (3) and using Leibnitz' rule we have

$$\begin{aligned}\frac{\delta}{\delta u^r}(FG) &= \sum_\nu (-1)^{|\nu|} D^\nu\left(F\frac{\partial G}{\partial u^r_\nu} + \frac{\partial F}{\partial u^r_\nu} G\right)\\ &= \sum_\nu (-1)^{|\nu|} \sum_{\alpha \leqslant \nu} \binom{\nu}{\alpha}\left(D^\alpha F.D^{\nu-\alpha}\frac{\partial G}{\partial u^r_\nu} + D^{\nu-\alpha}\frac{\partial F}{\partial u^r_\nu}.D^\alpha G\right)\\ &= \sum_\alpha \left(D^\alpha F.\sum_{\nu \geqslant \alpha}(-1)^{|\nu|}\binom{\nu}{\alpha} D^{\nu-\alpha}\frac{\partial G}{\partial u^r_\nu} + D^\alpha G \sum_{\nu\geqslant\alpha}(-1)^{|\nu|}\binom{\nu}{\alpha}D^{\nu-\alpha}\frac{\partial F}{\partial u^r_\nu}\right). \end{aligned} \quad (7)$$

If we change the variable ν by $\lambda = \nu - \alpha$, the conclusion follows at once.

Given m functions $\mu^r = \mu^r[x, u]$ of $\mathcal{F}$, the first-order differential operator

$$X_\mu = \sum_\alpha D^\alpha \mu^r.\frac{\partial}{\partial u^r_\alpha} \quad (8)$$

will be called the Lie–Bäcklund operator associated with the set of functions $\mu = (\mu^1, \ldots, \mu^m)$. This type of operator has been widely used in the analysis of the invariance groups of differential equations. These operators commute with the derivation operators D^α and form a Lie algebra (see the Appendix). We note that (4) implies that

$$X_\mu F = \sum_\alpha D^\alpha\left(\mu^r.\frac{\delta F}{\delta u^r_\alpha}\right). \quad (9)$$

The next proposition provides the commutation relations between the usual variational derivatives and Lie–Bäcklund operators.

Proposition 2.

$$\left[\frac{\delta}{\delta u^r}, X_\mu\right] = \sum_\alpha (-1)^{|\alpha|}\frac{\delta \mu^s}{\delta u^r_\alpha} D^\alpha \frac{\delta}{\delta u^s}. \quad (10)$$

Proof. From (6), (9) and proposition 1 we deduce that

$$\frac{\delta}{\delta u^r} X_\mu F = \frac{\delta}{\delta u^r}\left(\mu^s \frac{\delta F}{\delta u^s}\right) = \sum_\alpha (-1)^{|\alpha|}\frac{\delta \mu^s}{\delta u^r_\alpha} D^\alpha \frac{\delta F}{\delta u^s} + \sum_\alpha (-1)^{|\alpha|} D^\alpha \mu^s.\frac{\delta}{\delta u^r_\alpha}\left(\frac{\delta F}{\delta u^s}\right).$$

Then, we have only to prove that

$$(-1)^{|\alpha|}\frac{\delta}{\delta u^r_\alpha}\left(\frac{\delta F}{\delta u^s}\right) = \frac{\partial}{\partial u^s_\alpha}\left(\frac{\delta F}{\delta u^r}\right). \quad (11)$$

To do it, let us note that, given functions $\psi^r = \psi^r(x)$, $[\partial/\partial u^r_\alpha, X_\psi] = 0$ and therefore

$$X_\psi \frac{\delta}{\delta u^r} F = \sum_\alpha (-1)^{|\alpha|} D^\alpha \frac{\partial}{\partial u^r_\alpha} X_\psi F = \frac{\delta}{\delta u^r} X_\psi F.$$

But this implies

$$\sum_\alpha D^\alpha \psi^s \cdot \frac{\partial}{\partial u^s_\alpha}\left(\frac{\delta F}{\delta u^r}\right) = \frac{\delta}{\delta u^r}\left(\psi^s(x)\frac{\delta F}{\delta u^s}\right) = \sum_\alpha (-1)^{|\alpha|} D^\alpha \psi^s \cdot \frac{\delta}{\delta u^r_\alpha}\left(\frac{\delta F}{\delta u^s}\right).$$

This identity holds for arbitrary functions $\psi^s = \psi^s(x)$. Identifying coefficients of the derivatives $D^\alpha \psi^s$ we obtain (11) and therefore the conclusion follows.

3. Invariance groups and conservation laws

Let $\omega^r = \omega^r[x, u]$ $(r = 1, \ldots, m)$ be m independent functions in $\mathscr{F}$, and let us consider the system of differential equations

$$\omega^r[x, u] = 0. \tag{12}$$

These equations together with their differential consequences define a formal manifold Ω (Ibragimov 1976) in the space whose elements are the points with coordinates (x_i, u^r_α) $(i = 1, \ldots, n;\ r = 1, \ldots, m;\ |\alpha| \geq 0)$; this manifold is given by the equations

$$D^\alpha \omega^r[x, u] = 0, \qquad r = 1, \ldots, m, \quad |\alpha| \geq 0. \tag{13}$$

We denote by $\mathscr{F}(\Omega)$ the set of functions $F \in \mathscr{F}$ which vanish on the manifold Ω. Given $F, G \in \mathscr{F}$ we shall write $F \doteq G$ when $F - G \in \mathscr{F}(\Omega)$. In particular, we shall write $F \doteq 0$ when $F \in \mathscr{F}(\Omega)$.

Given functions ξ_i, $\eta^r \in \mathscr{F}$ $(i = 1, \ldots, n;\ r = 1, \ldots, m)$, let us consider the infinitesimal transformation group

$$x'_i = x_i + \epsilon \xi_i[x, u(x)], \qquad u'^r(x') = u^r(x) + \epsilon \eta^r[x, u(x)]. \tag{14}$$

To first order in the parameter ϵ, the total variation of the field functions is given by

$$u'^r(x) = u^r(x) + \epsilon \mu^r[x, u(x)], \qquad \mu^r = \eta^r - \boldsymbol{\xi} \cdot \boldsymbol{D} u^r. \tag{15}$$

The criterion for invariance of the system (12) with respect to the infinitesimal transformation group (14) is given by the condition (Ibragimov 1976, 1977)

$$\sum_i \xi_i \frac{\partial \omega^r}{\partial x_i} + \sum_\alpha \zeta^s_\alpha \frac{\partial \omega^r}{\partial u^s_\alpha} \doteq 0, \qquad r = 1, \ldots, m, \tag{16}$$

where

$$\zeta^r_\alpha = D^\alpha \mu^r + \boldsymbol{\xi} \cdot \boldsymbol{D} u^r_\alpha.$$

Evidently, the condition (16) is equivalent to

$$\boldsymbol{\xi} \cdot \boldsymbol{D} \omega^r + \sum_\alpha D^\alpha \mu^s \cdot \frac{\partial \omega^r}{\partial u^s_\alpha} \doteq 0, \qquad r = 1, \ldots, m. \tag{17}$$

From (8), and since $\boldsymbol{\xi} \cdot \boldsymbol{D} \omega^r \doteq 0$, we conclude that the invariance condition reduces to $X_\mu \omega^r \doteq 0$ for all $r = 1, \ldots, m$. We note that it depends only on the functions μ^r which determine the total variations of the fields. Then, we may adopt the following definition:

Definition. X_μ is said to be an invariance operator for the system (12) if $X_\mu \omega^r \doteq 0$ for all $r = 1, \ldots, m$.

We now turn our attention to the conservation laws.

Definition. By a conserved current of the system (12) we shall mean a vector function $\boldsymbol{A} = (A_1, \ldots, A_n)(A_i \in \mathscr{F})$ such that $\boldsymbol{D}\cdot\boldsymbol{A} \doteq 0$. A conservation law $\boldsymbol{A}$ will be called trivial if $\boldsymbol{A} \doteq 0$ or $\boldsymbol{D}\cdot\boldsymbol{A} = 0$.

When one of the coordinates x_i plays the role of time, then the time components of conserved currents are called conserved densities. Under appropriate decay properties the space integral of a conserved density becomes a constant of the motion.

An important class of conserved currents is provided by the vector functions $\boldsymbol{A}$ for which there are functions $\mu^r \in \mathscr{F}(r = 1, \ldots m)$ verifying

$$\boldsymbol{D}\cdot\boldsymbol{A} = \mu^r \omega^r. \tag{18}$$

They are called first-kind conserved currents. From (6), we deduce that the functions $\mu = (\mu^1, \ldots, \mu^m)$ associated with first-kind conserved currents are the solutions of the following system of equations:

$$\delta(\mu^s \omega^s)/\delta u^r = 0, \qquad r = 1, \ldots, m. \tag{19}$$

Let us consider evolution equations of the form

$$\partial u^r/\partial t - \sigma^r[x, u] = 0, \qquad r = 1, \ldots, m, \tag{20}$$

where $[x, u]$ denotes a finite subset of the set of variables (x_i, u^r_α), u^r_α $(r = 1, \ldots, m;$ $|\alpha| \geqslant 0)$ being arbitrary-order derivatives of the field functions with respect to the space coordinates $x_i(i = 1, \ldots, n)$. For these systems of equations it is unnecessary to consider conserved currents depending on the time derivatives of the fields because (20) allows us to write this type of derivative in terms of the remaining variables. On the other hand, given $F = F[t, x, u]$, we have

$$\frac{\mathrm{d}F}{\mathrm{d}t} \doteq \frac{\partial F}{\partial t} + \sum_\alpha \frac{\partial F}{\partial u^r_\alpha} D^\alpha \sigma^r = \frac{\partial F}{\partial t} + X_\sigma F. \tag{21}$$

Clearly F is a conserved density if and only if the right-hand side of (21) is a space divergence, but this holds if and only if

$$\frac{\delta}{\delta u^r}\left(\frac{\partial F}{\partial t} + X_\sigma F\right) = 0, \qquad r = 1, \ldots, m. \tag{22}$$

In the following two sections we shall see how the identities (7), (9) and (10) can be used to analyse the relationship between invariance groups and conservation laws in the Lagragian and Hamiltonian formalism.

4. Lagrangian formalism and Noether's transformations

We shall consider the case in which the functions ω^r coincide with the variational derivatives of a certain function $L \in \mathscr{F}$. Then (12) reduces to an Euler–Lagrange system of equations

$$\delta L/\delta u^r = 0. \tag{23}$$

From (19) we see that a set of functions $\mu=(\mu^1,\ldots,\mu^m)$ is associated with a first-kind conserved current of (23) if and only if

$$\frac{\delta}{\delta u^r}\left(\mu^s\frac{\delta L}{\delta u^s}\right)=0, \qquad r=1,\ldots,m. \tag{24}$$

However, if we consider the Lie–Bäcklund operator X_μ associated with μ, then (6) and (9) imply that (24) is equivalent to

$$\delta(X_\mu L)/\delta u^r=0, \qquad r=1,\ldots,m. \tag{25}$$

If X_μ satisfies this condition we will say that it is a Noether operator. In this case μ defines a conserved current, which will be denoted by $\boldsymbol{A}_\mu$, satisfying

$$\boldsymbol{D}\cdot\boldsymbol{A}_\mu=\mu^r\,\delta L/\delta u^r. \tag{26}$$

We note that (26) determines $\boldsymbol{A}_\mu$ up to a divergenceless term. Conversely, given a first-kind conserved current $\boldsymbol{A}_\mu$ verifying (26), then from the equivalence between (24) and (25) we deduce that X_μ is a Noether operator. Therefore, the solutions μ of (24) induce a one-to-one correspondence $X_\mu\to\boldsymbol{A}_\mu$ between Noether operators and first-kind conserved currents defined up to a divergenceless term. The following theorem establishes that this correspondence is in fact a correspondence between invariance operators and conserved currents.

Theorem 1. If X_μ is a Noether operator, then it is an invariance operator for the Euler–Lagrange equations.

Proof. From (10) we have

$$X_\mu\frac{\delta L}{\delta u^r}=\frac{\delta}{\delta u^r}X_\mu L-\sum_\alpha(-1)^{|\alpha|}\frac{\delta\mu^s}{\delta u^r_\alpha}D^\alpha\frac{\delta L}{\delta u^s}\stackrel{\circ}{=}\frac{\delta}{\delta u^r}X_\mu L.$$

Then, the conclusion follows from (25).

Usually Noether transformations are used in connection with the transformation properties of the action functional

$$W[u;V]=\int_V L[x,u]\,\mathrm{d}^n x \tag{27}$$

under an infinitesimal group like (14). It is easy to see that, to first order in the parameter ϵ,

$$W[u';V']-W[u;V]=\epsilon\int_V\delta L[x,u]\,\mathrm{d}^n x, \tag{28}$$

where

$$\delta L=\boldsymbol{D}\cdot(\boldsymbol{\xi}L)+X_\mu L, \qquad \mu^r=\eta^r-\boldsymbol{\xi}\cdot\boldsymbol{D}u^r. \tag{29}$$

Using (9), we find the following expression for δL:

$$\delta L=\boldsymbol{D}\cdot\boldsymbol{A}_0+\mu^r\,\delta L/\delta u^r, \tag{30}$$

where $\boldsymbol{A}_0$ is determined up to a divergenceless term by the equation

$$\boldsymbol{D}\cdot\boldsymbol{A}_0=\boldsymbol{D}\cdot(\boldsymbol{\xi}L)+\sum_{|\alpha|\geq 1}D^{\alpha}\left(\mu^{r}\frac{\delta L}{\delta u_{\alpha}^{r}}\right). \tag{31}$$

There are quite different concepts of Noether transformations in the literature. A common formulation (Hill 1951, Steudel 1967) is to say that (14) defines a Noether transformation group if there is a vector function $\boldsymbol{K}[x, u]$ such that

$$\delta L=\boldsymbol{D}\cdot\boldsymbol{K}. \tag{32}$$

In this case (30) leads to the conserved current $\boldsymbol{A}_{\xi,\mu}=\boldsymbol{K}-\boldsymbol{A}_0$. From (29) it is clear that (32) is equivalent to (25). Therefore they define the same type of Noether transformations. From the equivalence between both conditions it follows that the functions ξ_i must be irrelevant in the condition (32) as well as in the form of the conserved current $\boldsymbol{A}_{\xi,\mu}$. In fact, if we substitute $\boldsymbol{\xi}$ by $\boldsymbol{\xi}'$ and the functions μ^r remain fixed, then by (29) δL is again a divergence $\boldsymbol{D}\cdot\boldsymbol{K}'=\boldsymbol{D}\cdot\boldsymbol{K}+\boldsymbol{D}\cdot[(\boldsymbol{\xi}-\boldsymbol{\xi}')L]$, and we have

$$\boldsymbol{D}\cdot\boldsymbol{A}_{\xi',\mu}=\boldsymbol{D}\cdot\boldsymbol{A}_{\xi,\mu}=\mu^{r}\,\delta L/\delta u^{r}.$$

This shows that both conserved currents coincide up to a trivial divergenceless term with the first-kind conserved current associated with the Noether operator X_μ. Then we conclude that the correspondence between invariance groups and conservation laws which is implicit in the conventional formulation of Noether theorem is given by the map $X_\mu\to\boldsymbol{A}_\mu$ between Noether's operators and first-kind conserved currents.

We emphasise two aspects of our approach to Noether's theorem. First, the redundant function $\boldsymbol{\xi}$ is not used; and, second, the invariance property of Euler–Lagrange equations under Noether transformations has been deduced in a simple way without reference to the transformation properties of the action functional.

On the other hand, the use of generalised variational derivatives permits us to obtain the conserved density associated with a given Noether operator. To see this, let us introduce the n dimensional vector basis $e_1=(1,0,\ldots,0),\ldots,e_n=(0,\ldots,0,1)$. We let

$$D^{ij\ldots}=D^{e_i+e_j+\ldots},\qquad u_{ij\ldots}^{r}=D^{ij\ldots}u^{r}.$$

For each α let $[\alpha]=\alpha_1!\ldots\alpha_n!/|\alpha|!$. In the same way $[ij\ldots]=[e_i+e_j+\ldots]$. With this notation, given a solution μ of (24), then from (5) we find that the components of $\boldsymbol{A}_\mu$ can be written in the form

$$A_i[x,u]=a_i(x)+\sum_{k\geq 0}[ii_1\ldots i_k]D^{i_1\cdots i_k}\int_{\gamma}\mathrm{d}v^{r}\frac{\delta}{\delta v_{ii_1\ldots i_k}^{r}}\left(\mu^{s}\frac{\delta L}{\delta v^{s}}\right), \tag{33}$$

where $\boldsymbol{a}(x)$ is a solution of $\boldsymbol{D}\cdot\boldsymbol{a}(x)=(\mu^{s}\delta L/\delta u^{s})[x,0]$ and γ is the curve $\gamma(t)=[x,tu]$ ($t\in[0,1]$). Another expression for $\boldsymbol{A}_\mu$ can be deduced by means of condition (32) to characterise Noether operators. In this way, given a vector function $\boldsymbol{K}$ such that $\delta L=\boldsymbol{D}\cdot\boldsymbol{K}$, then from (30) and (31) we obtain

$$A_i[x,u]=K_i-\xi_iL-\sum_{k\geq 0}[ii_1\ldots i_k]D^{i_2\cdots i_k}\left(\mu^{s}\frac{\delta L}{\delta u_{ii_1\ldots i_k}^{s}}\right). \tag{34}$$

This latter formula is useful for practical purposes when we are able to find the function $\boldsymbol{K}$. For instance, if the Lagrangian function L does not depend explicitly on the coordinates x_j, then if we take $\boldsymbol{\xi}$ arbitrary and $\mu^r = u_j^r$ we have

$$\delta L = \boldsymbol{D} \cdot (\boldsymbol{\xi} L) + X_\mu L = \boldsymbol{D} \cdot (\boldsymbol{\xi} L) + D^j L.$$

Therefore $\delta L = \boldsymbol{D} \cdot \boldsymbol{K}$ where $K_i = \xi_i L + \delta_{ij} L$ and (34) provides the following formula for the conserved energy–momentum tensor:

$$T_{ij} = \sum_{k \geq 0} [ii_1 \ldots i_k] D^{i_1 \cdots i_k} \left(u_j^s \frac{\delta L}{\delta u^s_{ii_1 \ldots i_k}} \right) - \delta_{ij} L. \tag{35}$$

5. Hamiltonian formalism and canonical operators

Let $\mathscr{H}$ be the set of functions in C^∞ depending on $n+1$ variables (t, x_i) $(i = 1, \ldots, n)$, $2m$ fields (ϕ^r, π^r) $(r = 1, \ldots, m)$ and a finite number of arbitrary-order derivatives $(\phi^r_\alpha, \pi^r_\alpha)$ $(|\alpha| \geq 0)$ with respect to the variables x_i. Given $F \in \mathscr{H}$ we consider the operator

$$U_F = \sum_\alpha \left(D^\alpha \left(-\frac{\delta F}{\delta \pi^r} \right) \frac{\partial}{\partial \phi^r_\alpha} + D^\alpha \left(\frac{\delta F}{\delta \phi^r} \right) \frac{\partial}{\partial \pi^r_\alpha} \right).$$

Clearly U_F coincides with the Lie–Bäcklund operator X_μ where $\mu = (-\delta F/\delta \pi^r, \delta F/\delta \phi^r)$ $(r = 1, \ldots, m)$. Then (9) implies that

$$U_F = \sum_\alpha D^\alpha \left(\frac{\delta F}{\delta \phi^r} \cdot \frac{\delta}{\delta \pi^r_\alpha} - \frac{\delta F}{\delta \pi^r} \frac{\delta}{\delta \phi^r_\alpha} \right). \tag{36}$$

In what follows, U_F will be called a canonical operator. The following proposition gives the commutation properties of canonical operators with variational derivatives.

Proposition 3. Given $F, G \in \mathscr{H}$, we have

$$[U_F, \delta/\delta\phi^r] G = U_G \, \delta F/\delta \phi^r, \tag{37a}$$

$$[U_F, \delta/\delta\pi^r] G = U_G \, \delta F/\delta \pi^r. \tag{37b}$$

Proof. Since U_F is the Lie–Bäcklund operator X_μ associated with $\mu = (-\delta F/\delta \pi^r, \delta F/\delta \phi^r)$, from (10) we obtain

$$\left[U_F, \frac{\delta}{\delta \phi^r} \right] G = \sum_\alpha (-1)^{|\alpha|} \left(\frac{\delta}{\delta \phi^r_\alpha} \left(\frac{\delta F}{\delta \pi^s} \right) D^\alpha \frac{\delta G}{\delta \phi^s} - \frac{\delta}{\delta \phi^r_\alpha} \left(\frac{\delta F}{\delta \phi^s} \right) D^\alpha \frac{\delta G}{\delta \pi^s} \right).$$

Identity (7) implies

$$\left[U_F, \frac{\delta}{\delta \phi^r} \right] G - \left[U_G, \frac{\delta}{\delta \phi^r} \right] F = \frac{\delta}{\delta \phi^r} \left(\frac{\delta G}{\delta \phi^s} \frac{\delta F}{\delta \pi^s} - \frac{\delta G}{\delta \pi^s} \frac{\delta F}{\delta \phi^s} \right).$$

Equation (36) leads to

$$\left[U_F, \frac{\delta}{\delta \phi^r} \right] G = \left[U_G, \frac{\delta}{\delta \phi^r} \right] F + \frac{\delta}{\delta \phi^r} U_G F = U_G \frac{\delta F}{\delta \phi^r}.$$

This proves (37a). The proof of (37b) is similar.

Given $H \in \mathscr{H}$, let us consider a Hamiltonian system of equations

$$\frac{\partial \phi^r}{\partial t} - \frac{\delta H}{\delta \pi^r} = 0, \qquad \frac{\partial \pi^r}{\partial t} + \frac{\delta H}{\delta \phi^r} = 0. \tag{38}$$

In this case equation (21) becomes

$$\mathrm{d}F/\mathrm{d}t \stackrel{\circ}{=} (\partial/\partial t - U_H)F. \tag{39}$$

If we introduce the operator on $\mathscr{H}$

$$D_t = \partial/\partial t - U_H,$$

then the system of equations (22) for the conserved densities reduces to

$$\frac{\delta}{\delta \phi^r} D_t F = \frac{\delta}{\delta \pi^r} D_t F = 0, \qquad r = 1, \ldots, m. \tag{40}$$

It is clear that if F is a spatial divergence, then it defines a trivial conserved density.

Given $F \in \mathscr{H}$ we consider the infinitesimal transformation group

$$\phi'^r(t, x) = \phi^r(t, x) - \epsilon\, \delta F/\delta \pi^r, \qquad \pi'^r(t, x) = \pi^r(t, x) + \epsilon\, \delta F/\delta \phi^r, \tag{41}$$

and its corresponding action on the Hamiltonian equations (38). Evidently the action of this group on the space $\mathscr{H}$ is generated by the canonical operator U_F. We shall denote by $\hat{U}_F$ the extension of U_F which acts on functions depending on time derivatives of the fields as in the equations (38).

Proposition 4. The operator $\hat{U}_F$ satisfies

$$\hat{U}_F\left(\frac{\partial \phi^r}{\partial t} - \frac{\delta H}{\delta \pi^r}\right) \stackrel{\circ}{=} -\frac{\delta}{\delta \pi^r} D_t F, \tag{42a}$$

$$\hat{U}_F\left(\frac{\partial \pi^r}{\partial t} + \frac{\delta H}{\delta \phi^r}\right) \stackrel{\circ}{=} \frac{\delta}{\delta \phi^r} D_t F. \tag{42b}$$

Proof. We have

$$\hat{U}_F\left(\frac{\partial \phi^r}{\partial t} - \frac{\delta H}{\delta \pi^r}\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta F}{\delta \pi^r}\right) - U_F \frac{\delta H}{\delta \pi^r} \stackrel{\circ}{=} -\frac{\delta}{\delta \pi^r}\left(\frac{\partial F}{\partial t}\right) + U_H \frac{\delta F}{\delta \pi^r} - U_F \frac{\delta H}{\delta \pi^r}.$$

From (37b) we obtain

$$\hat{U}_F\left(\frac{\partial \phi^r}{\partial t} - \frac{\delta H}{\delta \pi^r}\right) \stackrel{\circ}{=} -\frac{\delta}{\delta \pi^r}\left(\frac{\partial F}{\partial t} - U_H F\right),$$

which proves (42a). The proof of (42b) is identical.

This proposition allows us to give a simple proof of a result due to Kumei (1978) which states the correspondence between invariance canonical operators and conserved densities for Hamiltonian systems in field theory.

Theorem 2. $\hat{U}_F$ is an invariance operator for Hamilton's equations if and only if F is a conserved density.

Proof. By (42*a*) and (42*b*) it is clear that $\hat{U}_F$ is an invariance operator if and only if

$$\frac{\delta}{\delta\phi^r} D_t F \doteq 0, \qquad \frac{\delta}{\delta\pi^r} D_t F \doteq 0. \tag{43}$$

But the functions $\delta(D_t F)/\delta\phi^r$ and $\delta(D_t F)/\delta\pi^r$ do not depend on the time derivatives of the fields (ϕ^r, π^r). Therefore (43) is equivalent to

$$\frac{\delta}{\delta\phi^r} D_t F = \frac{\delta}{\delta\pi^r} D_t F = 0.$$

Then the theorem is proved.

Since two operators $\hat{U}_F$ and $\hat{U}_{F'}$ are equal if and only if $F - F'$ is a divergence term, theorem 2 tells us that the correspondence $\hat{U}_F \to F$ is a one-to-one mapping between invariance canonical operators and conserved densities defined up to trivial divergence terms.

We note that our derivation of theorem 2 uses only algebraic properties between variational derivatives and canonical operators and has been carried out within the context of density functions. However, in order to obtain a constant of motion from a given conserved density F, we must give a meaning to the space integral of F, and a space integral of a density function of the type of a divergence must vanish. It is required to impose boundary conditions for the field functions. In fact they are also necessary to formulate the initial-value problem of Hamilton's equations. But the property implicit in theorem 2 is purely algebraic and therefore its derivation does not require assumptions about boundary conditions. This fact appears clearly in our approach but it is not obvious in Kumei's paper (1978) in which the use of functionals removes the problem from its algebraic context.

6. Concluding remarks

In this paper several identities involving generalised variational derivatives and Lie–Bäcklund operators have been deduced which provide a new approach to conservation theorems in Lagrangian and Hamiltonian formalisms. In this way, it is shown that the correspondence between symmetries and conservation laws which is implicit in Noether's theorem can be derived without reference to the transformation properties of the action functional, and a simple proof of the fact that Noether operators are invariance Lie–Bäcklund operators for Euler–Lagrange equations is given. Moreover, the use of generalised variational derivatives allows us to write in a closed form the Noether currents arising in the case of Lagrangian densities containing higher-order derivatives. In Hamiltonian formalism a proof of Kumei's theorem (1978) has been achieved on purely algebraic considerations. Boundary conditions for field functions had not to be taken into account in proving this theorem. These results, together with those of an earlier publication (Galindo and Martínez Alonso 1978), reveal that several important questions of the calculus of variations admit a rigorous formulation by means of algebraic methods.

Finally let us mention other possible applications of the present approach. From identity (10), the system (22) which characterises the conserved densities of an evolution equation can be rewritten as

$$\frac{\partial\lambda^r}{\partial t}+\sum_\alpha\left(D^\alpha\sigma^r\cdot\frac{\partial\lambda^r}{\partial u^r_\alpha}+(-1)^{|\alpha|}\frac{\delta\sigma^s}{\delta u^r_\alpha}D^\alpha\lambda^r\right)=0, \tag{44}$$

where $\lambda^r=\delta F/\delta u^r (r=1,\ldots,m)$. This linear system of partial differential equations for the unknown functions λ^r can be used to find conserved densities of arbitrary evolution equations. To illustrate briefly this method, let us consider the equation

$$u_t-(u^p)_{xx}=0, \tag{45}$$

where p is a real number different from zero. For $p\geqslant 2$ it describes the density of a gas flowing through a homogeneous porous medium (Muskat 1937). In this case (44) reduces to

$$\frac{\partial\lambda}{\partial t}+pu^{p-1}D^2_x\lambda+\sum_n\frac{\partial\lambda}{\partial u_n}D^{n+2}_x u^p=0. \tag{46}$$

Let us suppose that λ depends on the derivatives $u_n=\partial^n u/\partial x^n$ up to a maximal order $N\geqslant 0$. Then, if we differentiate (46) with respect to the variable u_{N+2}, we obtain

$$2pu^{p-1}\,\partial\lambda/\partial u_N=0.$$

This implies $\lambda=\lambda(t,x)$, and (45) becomes

$$\partial\lambda/\partial t+pu^{p-1}\,\partial^2\lambda/\partial x^2=0.$$

Then, up to a total derivative term, the conserved densities of (45) are

$$p\neq 1:\qquad F=(ax+b)u,\qquad a,b\in\mathbb{R};$$

$$p=1:\qquad F=\psi(t,x)u,\qquad \psi/\partial_t\psi+\partial_{xx}\psi=0.$$

Work is in progress to apply this kind of technique to several evolution equations of classical physics.

Appendix

We consider first-order differential operators of the form

$$X=\sum_\alpha \eta^r_\alpha\cdot\frac{\partial}{\partial u^r_\alpha}, \tag{A1}$$

where η^r_α are functions in C^∞ in the variables x_i, $u^r_\beta (i=1,\ldots,n; r=1,\ldots,m; |\beta|\geqslant 0)$. Given two of these operators, X and $\bar{X}$, associated with functions η^r_α and $\bar{\eta}^r_\alpha$ respectively, their commutator is given by

$$[X,\bar{X}]=\sum_\alpha(X\bar{\eta}^r_\alpha-\bar{X}\eta^r_\alpha)\frac{\partial}{\partial u^r_\alpha}. \tag{A2}$$

This expression allows us to calculate the commutator of an operator X with the operator D^{e_i} of total differentiation with respect to the variable x_i:

$$[X,D^{e_i}]=\sum_\alpha(\eta^r_{\alpha+e_i}-D^{e_i}\eta^r_\alpha)\frac{\partial}{\partial u^r_\alpha}. \tag{A3}$$

Therefore it is clear that the operators X which commute with the operators D^{e_i} are those whose associated functions η^r_α are of the form $\eta^r_\alpha = D^\alpha \eta^r$, that is, they are the Lie–Bäcklund operators.

From (A2) we deduce that

$$[X_\eta, X_\xi] = \sum_\alpha D^\alpha (X_\eta \xi^r - X_\xi \eta^r) \frac{\partial}{\partial u^r_\alpha}, \tag{A4}$$

i.e.

$$[X_\eta, X_\xi] = X_{[\eta, \xi]}, \tag{A5}$$

where

$$[\eta, \xi]^r = X_\eta \xi^r - X_\xi \eta^r. \tag{A6}$$

Therefore Lie–Bäcklund operators form a Lie algebra.

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